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Noncontinuous Minkowskian Spacetime

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A model for a noncontinuous spacetime is considered, which is based on a generalization of the manifold concept, known as d-space. As a consequence of its construction, the model possesses a metric of Lorentzian signature and a generalized form of Lorentz invariance. The model may be considered as defining a class of discrete spacetimes within the framework of the d-space representation of general relativity.

1. INTRODUCTION

Spacetime is conventionally based on the manifold concept and the requirement of a metric of Lorentzian signature. Though intuitive, the assumption of universal continuity is debatable. Applying quantum mechanical arguments, it seems natural to include the possibility that at microscopic scales of the order of the Planck scale spacetime may become discrete. Historically, the idea that space could become discrete goes back to Riemann (1919), who reflected about a natural measure of space. Attempts to consider spacetime discrete have subsequently been undertaken in order to investigate the effects of introducing a hypothetical constant length ℓ into the theory of elementary particles (Heisenberg, 1942), construct a discrete spacetime by extending it to five dimensions (Snyder, 1947), or, at the expense of violation of Lorentz invariance, replace Minkowski spacetime with a cellular model of spacetime (Das, 1960). More recently, it has been hypothesized (Bombelli *et al.*, 1987) that microscopic spacetime possesses the structure of a partially ordered set of points. These authors investigated the mathematical relation between such a set and a corresponding smooth manifold. It was demonstrated subsequently (Brightwell and Gregory, 1991) that if the point set is of random

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character, a certain smoothing (coarse graining) leads to the wanted manifold properties. The philosophy underlying these approaches culminates in the wish to root the classical continuous spacetime in a discrete one, having causal structure. The basic motivation for introducing discrete spacetimes arises from the conceptual problems of quantum gravity, singularities of general relativity, and the divergence problem in quantum field theories.

Recently it has been proposed (Gruszczak *et al.*, 1988; Heller, 1989) to identify spacetime with d-spaces in place of manifolds. The d-space concept (Sikorsky, 1992) generalizes the concept of manifolds by dropping the axiom which makes sure that the manifold is locally diffeomorphic to $\mathbf{R}^n$, retaining that large parts of differential geometry are applicable to d-spaces. Since every subset of $\mathbf{R}^n$ can be made into a d-space, the d-space concept provides a convenient tool to treat those subsets of $\mathbf{R}^n$ which cannot be identified with manifolds in analogy to manifolds. In order to find a representation of spacetimes which is of greater generality than manifold-based spacetimes, we follow the suggestion (Gruszczak *et al.*, 1988; Heller *et al.*, 1989) to seek for a possible representation of spacetime in terms of d-spaces. A requirement on such a representation (Gruszczak *et al.*, 1988) is the existence of a metric of Lorentzian signature on the d-space under consideration.

As done in Multarzynski and Heller (1992), the mathematical framework of the d-space concept may be used to treat the singularities of spacetimes based on the manifold concept. But it also provides a means for treating a class of entirely different spacetimes; in particular, spacetimes which are not diffeomorphic to $\mathbf{R}^n$. In this paper we focus on the latter problem by presenting a *Minkowskian* d-space. As consequence of its construction, the d-space possesses a metric of Lorentzian signature and a generalized form of Lorentz invariance. As will be outlined in the last section, the spacetime represented by the proposed d-space is a two-dimensional massless solution of a generalized form of the Einstein equations. Besides, it may be considered as defining a class of (not massless) solutions by the demand that its solutions are, according to the common structure of general relativity, locally diffeomorphic to the *Minkowskian* d-space under consideration.

2. THE d-SPACE FORMALISM

Let a smooth manifold be defined as a pair (M, C) where C is a family of real functions on an (abstract) set M, satisfying suitable axioms. These axioms can be summarized for short by (i) C is closed with respect to localization and (ii) C is closed with respect to composition of the set $\mathscr{E}$ of all C^∞-functions on $\mathbf{R}^n$. Axiom (iii) states that M is locally diffeomorphic to $\mathbf{R}^n$. It may be proven that the manifold definition by axioms (i)–(iii) and the one in the ordinary terms of equivalence classes of atlases are equivalent

(Gruszczak *et al.*, 1988). By dropping axiom (iii), one obtains a more general structure called a differential space or d-space. C is said to be the d-structure on the support M, and the pair (M, C) denotes the d-space. It may be shown (Gruszczak *et al.*, 1988) that for every family of functions C_0 inducing a topology T_{C_0} on M, there exists a smallest family of functions C defining a d-structure on M such that $C_0 \subset C$ and the topology T_{C_0} coincides with the topology T_C on M. The C_0 is said to generate the d-structure C of the d-space (M, C). According to the common interpretation (Gruszczak *et al.*, 1988; Heller *et al.*, 1989), the family C_0 represents the family of scalar fields which provides information about measurements on M, i.e., C_0 contains *all physical* information of (M, C).

3. CONSTRUCTION OF THE d-SPACE

In this section, the following idea is pursued: A subset M of $\mathbf{R}^2$ which is nowhere diffeomorphic to $\mathbf{R}^2$ and possesses certain invariance properties is transformed into a d-space in order to obtain a mathematically treatable representation.

Before transforming M into a d-space (M, C) it may be convenient to add some remarks about the mentioned invariance properties of M. First, let the subset M of $\mathbf{R}^2$ be defined by the following expression:

$$M = \left\{\frac{n}{m}(r^2 + s^2), \frac{n}{m}(r^2 - s^2)\right\} \subset \mathbf{R}^2 \tag{1}$$

with $r, s, n, m \in \mathbf{Z}\backslash\{0\}$. Second, consider the family of mappings ϕ defined by

$$\begin{aligned} \phi\colon\ & M \mapsto M \\ \phi(p) &= \mathbf{A}p| \end{aligned} \tag{2}$$

with $\mathbf{A} = \mathbf{A}(v) = \mathbf{A}(r, s)$ where, again with $r, s \in \mathbf{Z}\backslash\{0\}$,

$$\begin{aligned} v &= \frac{r^2 - s^2}{r^2 + s^2} \\ \mathbf{A}(v) &= \frac{1}{(1 - v^2)^{1/2}}\begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix} \\ &= \frac{1}{2rs}\begin{pmatrix} r^2 + s^2 & -(r^2 - s^2) \\ -(r^2 - s^2) & r^2 + s^2 \end{pmatrix} = \mathbf{A}(r, s) \end{aligned} \tag{3}$$

It may be easily shown that (i) the elements $\varphi \in \phi$ map onto M, that (ii) $(\phi, \circ)$ possesses group structure, and that (iii) $(\phi, \circ)$ is a subgroup of the common $(1 + 1)$-dimensional Lorentz group (using natural units). Together with the discreteness of M, these are the properties which let M be of interest for the representation of spacetime.

The transformation of M into a d-space (M, C) involves the definition of the family of mappings C_0 which generates the d-structure C of the d-space (M, C). Let the family C_0 be defined by

$$D_0 = \{\pi_i: \mathbf{R}^2 \to \mathbf{R}\}, \qquad i = 0, 1 \qquad (4)$$
$$\pi_i: \quad \mathbf{R}^2 \ni (x_0, x_1) \to x_i$$

together with the restriction

$$C_0 = D_0|_M \qquad (5)$$

As result of this construction, the topology T_{C_0} coincides with the topology on M induced from $\mathbf{R}^2$ and C_0 generates the d-structure $C = \mathscr{E}|_M$, where $\mathscr{E}$ denotes the set of all C^∞-functions on $\mathbf{R}^2$. The support M together with the d-structure C form the specific d-space (M, C). In analogy, $(\mathbf{R}^2, \mathscr{E})$ represents a d-space which is diffeomorphic to $\mathbf{R}^2$. Since $M \subset \mathbf{R}^2$ and $C_0 = D_0|_M$, (M, C) is a d-subspace of the d-space $(\mathbf{R}^2, \mathscr{E})$.

Most of the mathematics on d-spaces is developed in terms of tangent vectors and tangent spaces. Tangent vectors to (M, C) at $p \in M$ are linear mappings v: $C \to \mathbf{R}$ satisfying the Leibniz condition $v(fg) = v(f)g(p) + f(p)v(g)$. Since M is dense and $C_0 \subset \mathscr{E}$, the limes

$$h \in C, \qquad (r, x_1), (x_0, s), (r, s) \in M$$
$$\left.\frac{\partial h}{\partial x_0}\right|_{(r,s)} = \lim_{x_0 \to r} \frac{h(x_0, s) - h(r, s)}{x_0 - r} \qquad (6)$$
$$\left.\frac{\partial h}{\partial x_1}\right|_{(r,s)} = \lim_{x_1 \to s} \frac{h(r, x_1) - h(r, s)}{x_1 - s}$$

exist and the two mappings v_1 and v_2, defined by

$$v_i: \quad C \to \mathbf{R}, \qquad H \in C, \qquad i = 0, 1 \qquad (7)$$
$$v_i(h) = \left.\frac{\partial h}{\partial x}\right|_p$$

form two tangent vectors to (M, C) at $p \in M$. The set of all tangent vectors to (M, C) at p is called the tangent space to (M, C) at p and is denoted by M_p. Notice that the tangent space M_p is a vector space and, since v_1 and v_2 are linearly independent, v_1 and v_2 form a basis of M_p. Because the local dimension of (M, C), $\dim_p((M, C))$, defined as the dimension of the tangent space M_p at $p \in M$, is constant on M, the global dimension of (M, C) is

$\text{Dim}((M, C)) = \dim_p((M, C)) = 2$. This is an important result, since it allows us to define a scalar product on M_p and on (M, C), respectively.

Consider now the tangent vector fields V_1 and V_2 defined by

$$
\begin{aligned}
&V_i\colon \quad M \ni p \mapsto V_i(p) \in M_p \subset \cup_{q \in M} M_q, \qquad i = 0, 1 \\
&V_i(p) = \left.\frac{\partial}{\partial x_i}\right|_p
\end{aligned}
\tag{8}
$$

According to the d-space terminology, since the mapping

$$V_i(\bullet)(p)\colon \quad M \ni p \to V_i(p)(h) \in \mathbf{R}, \qquad i = 0, 1 \tag{9}$$

is an element of C for every $f \in C$, V_1 and V_2 are said to be smooth. The set $\mathcal{M}(M)$ of all smooth tangent vector fields on (M, C) is a module over C. Since V_1 and V_2 are linearly independent and $V_1(p)$ and $V_2(p)$ form a basis of M_p, the tangent vector fields V_1 and V_2 form a basis of $\mathcal{M}(M)$. The dual basis V^1 and V^2 of $\mathcal{M}(M)$ is given by the differentials dx_0 and dx_1, respectively.

A scalar product G on M_p and $\mathcal{M}(M)$, respectively, is defined by

$$
\begin{aligned}
&G\colon \quad \mathcal{M} \times \mathcal{M} \to C \\
&G(U, W) = U^1W_1 - U^2W_2
\end{aligned}
\tag{10}
$$

with $U, W \in \mathcal{M}(M)$. With respect to the scalar product G, $V_1(p)$ and $V_2(p)$ form an orthonormal vector basis of M_p, while V_1 and V_2 form one of $\mathcal{M}(M)$. The index of $\mathcal{M}(M)$, defined as number of minus signs in the expression

$$G(V_i, V_i) = \epsilon_i = \pm 1, \qquad i = 0, 1 \tag{11}$$

is 1. Hence, $\mathcal{M}(M)$ is a Lorentz module on (M, C), and the triple (M, C, G) is a Lorentz d-space.

Two d-spaces (M, C) and (N, D) are said to be diffeomorphic if the mappings ψ, ψ^{-1} exist such that $\psi\colon M \mapsto N$ and $f \circ \psi \in C$ for every $f \in D$. Thus, according to the d-space terminology, the family of mappings ϕ defined in (3) and interpreted as

$$\phi\colon \quad (M, C) \mapsto (M, C) \tag{12}$$

is a group $(\phi, \circ)$ of diffeomorphisms onto (M, C). Since, as may be shown by direct calculation, the mappings $\phi \ni \varphi\colon (M, C) \mapsto (M, C)$ preserve the scalar product G, $(\phi, \circ)$ establishes a group of isometric diffeomorphisms onto M. Therefore, the Lorentzian d-space (M, C, G) possesses two important properties of special relativity: (i) G is a scalar product (or metric) of Lorentzian signature, and (ii) the family ϕ establishes a group of isometric diffeomorphisms onto (M, C, G). Besides this invariance of (M, C, G) under the group $(\phi, \circ)$ there exists another link between the d-space (M, C, G) and the family ϕ, since the parameter range of the family ϕ is definable by the range of the function $v \in C$,

$$v\colon M \to \mathbf{R}, \qquad v(x_0, x_1) = \frac{x_1}{x_0}$$

One may interpret the Lorentz d-space (M, C, G) as a complete d-space analog to a $(1 + 1)$-dimensional subset of the ordinary Minkowski spacetime. This property suggests that we denote the d-space (M, C, G) as *Minkowskian* d-space and to consider the invariance of (M, C, G) and other entities under the group $(\phi, \circ)$ as a generalized form of two-dimensional Lorentz invariance.

The straightforward extension of the d-space (M, C, G) to a four-dimensional d-space (M', C', G') which preserves the essential properties of (M, C, G) proceeds by reinterpreting, geometrically speaking, x_1 in the set M as a radial coordinate in a spherical, three-dimensional representation. By this procedure M is extended to a four-dimensional set M'. Such an extension resembles the transition from two to four dimensions in special relativity, where it is achieved by considering spatial rotations. The family C_0 will also have to be extended, and the family of diffeomorphisms ϕ would incorporate rotations in the coordinates x_1, x_2, x_3. Such an extension will, however, not lead to any new essential properties besides providing by the d-space (M', C', G') an analog to the four-dimensional Minkowski spacetime.

To finish this section let us briefly consider the d-subspace $(\partial M, \mathscr{C}|_{\partial M})$ of $(\mathbf{R}^2, \mathscr{C})$. In view of the completion of (M, C) in order to achieve a d-space (M', C'), which is diffeomorphic to $(\mathbf{R}^2, \mathscr{C})$, $(\partial M, \mathscr{C}|_{\partial M})$ represents the boundary of (M, C). Since the boundary $(\partial M, \mathscr{C}|_{\partial M})$ and (M, C) are d-subspaces of $(\mathbf{R}^2, \mathscr{C})$, according to a common classification scheme (Ellis and Schmidt, 1977), the boundary is of regular character. In other words, the singularities of (M, C) which prevent (M, C) from being diffeomorphic to $(\mathbf{R}^2, \mathscr{C})$ are removable; the d-space (M, C) may be extended to a d-space which is diffeomorphic to $(\mathbf{R}^2, \mathscr{C})$ without changing its topology.

4. CONCLUSIONS

In the previous section, a Minkowskian d-space (M, C, G) was constructed. Here we infer its physical content and possible interpretation. According to the properties of (M, C, G) discussed above, the d-space (M, C, G) corresponds to an equivalence class of manifolds or coordinate systems which is identifiable with the subclass of the reference frames of special relativity. Because the Einstein equations can be formulated within the d-space formalism (Heller *et al.*, 1989), the d-space (M, C, G) may also be interpreted in the context of general relativity. On making the transition from manifolds to d-spaces, the set of solutions of the Einstein equations is generalized to include solutions which are not diffeomorphic to $\mathbf{R}^n$. In this sense, the d-space (M, C, G) represents a two-dimensional massless solution of a generalized form of Einstein's vacuum field equations (Heller *et al.*, 1989)

$$R^v = 0$$

where R^v is the Ricci tensor. (M, C, G) is *not* diffeomorphic to $\mathbf{R}^2$. The most striking implication therefore is the lack of general validity of Einstein's equivalence principle, as in the foregoing discussion axiom (iii) has nowhere been made use of. However, it seems that at least in the case of the restricted model under consideration, axiom (iii) is not needed in order to establish the basic structures of special and general relativity.

In full analogy to the common structure of special and general relativity, one may demand for other (not massless) solutions of the generalized form of the Einstein equations that they are locally diffeomorphic to (M, C, G). This demand characterizes a class of spacetimes which are nowhere diffeomorphic to $\mathbf{R}^2$, but do locally possess properties of macrophysical spacetime.

Summarizing, a $(1 + 1)$-dimensional spacetime model has been constructed which is nowhere diffeomorphic to $\mathbf{R}^2$ but possesses structures of the common $(1 + 1)$-dimensional Minkowski spacetime. The model in its d-space representation (M, C, G) is mathematically treatable. As has been shown, the d-space (M, C, G) may be considered as defining a class of discrete spacetimes within the framework of a generalized form of general relativity, which without being processed by a coarse graining possesses properties familiar from macrophysical spacetime. However, at this stage, the question of significance of those spacetimes remains open. In particular, since (M, C, G) is dense, one probably cannot hope these spacetimes to cure the divergence problem of quantum field theories.

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